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LETTER TO THE EDITOR

Dicke's model phase transition, when counter-rotating and $(A)^2$ terms are included in the Hamiltonian

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Abstract. The $(A)^2$ term does not affect the phase transition. If counter-rotating terms are included in the model, the phase transition persists with a renormalized coupling constant.

Since the discovery of a phase transition present in Dicke's model for a maser (Hepp and Lieb 1973), the interest in the model was renewed. This phase transition is characterized by an instability of the ground state (Narducci *et al* 1973).

It is shown in this letter, that the A^2 term in the Hamiltonian does not affect the phase transition and that the presence of the counter-rotating terms introduce a renormalization of the coupling constant.

Consider a Hamiltonian of the form:

$$H=H_0+H_1, \tag{1}$$

where H_1 is regarded as a perturbation. Making use of Bogoliubov's method (Bogoliubov *et al* 1957), a thermodynamically equivalent Hamiltonian is found. This Hamiltonian has the advantage over Dicke's that the bilinear products of spin and field operators are disposed of.

Considering the Hamiltonian of equation (1), it is simple to prove that:

$$\frac{\mathrm{Tr}\{\exp[-\beta(H_0+H_1)]\}}{\mathrm{Tr}[\exp(-\beta H_0)]}=1-\int_0^\beta \mathrm{d}\beta'\langle H_1\rangle_0+\int_0^\beta\int_0^{\beta'}\mathrm{d}\beta'\,\mathrm{d}\beta''\langle H_1^2\rangle_0-\ldots, \tag{2}$$

where:

$$\langle (H_1)^n\rangle_0=\frac{\mathrm{Tr}[(H_1)^n\exp(-\beta H_0)]}{\mathrm{Tr}[\exp(-\beta H_0)]}.$$

The idea is to find a thermodynamically equivalent Hamiltonian H_0, such that in the limit $N\to\infty$, the corrections in the partition function are minimized.

Consider Dicke's model with the A^2 term included:

$$H=a^\dagger a+\frac{\epsilon}{2}\sum_{i=1}^N\sigma_i^z+\frac{\lambda}{2\sqrt{N}}\sum_{i=1}^N(a\sigma_i^+ +a^\dagger\sigma_i^-)+K(a+a^\dagger)^2. \tag{3}$$

Define the following linearizing transformations (Gibberd 1974, Gilmore and Bowden 1976):

$$\begin{aligned} a&=b+N^{1/2}\alpha\\ a^\dagger&=b^\dagger+N^{1/2}\alpha^*. \end{aligned} \tag{4}$$

Then we can write:

$$H_0 = b^\dagger b(2K+1) + |\alpha|^2 N + KN[2|\alpha|^2 + (\alpha)^2 + (\alpha^*)^2] + \tfrac{1}{2}\epsilon \sum_{i=1}^{N} \sigma_i^z + \tfrac{1}{2}\lambda \sum_{i=1}^{N} (\alpha\sigma_i^+ + \alpha^*\sigma_i^-), \tag{5}$$

$$H_1 = K[(b)^2 + (b^*)^2] + \left[b\left(N^{1/2}[(\alpha^*) + 2K(\alpha+\alpha^*)] + N^{-1/2}(\tfrac{1}{2}\lambda) \sum_{i=1}^{N} \sigma_i^+ \right) + \mathrm{cc} \right],$$

where cc stands for complex conjugate. The partition function Z, when the system is in thermal equilibrium, is:

$$Z = \mathrm{Tr}\, e^{-\beta H} \approx \mathrm{Tr}\, e^{-\beta H_0} = \frac{\exp[\![-\beta N\{|\alpha|^2 + K[2|\alpha|^2 + (\alpha)^2 + (\alpha^*)^2]\}]\!]}{1 - \exp[-\beta(2K+1)]} \prod_{i=1}^{N} \mathrm{Tr}\, e^{-\beta h_i}, \tag{6}$$

where:

$$h_i = \tfrac{1}{2}\epsilon \sum_{i=1}^{N} \sigma_i^z + \tfrac{1}{2}\lambda \sum_{i=1}^{N} (\alpha\sigma_i^+ + \alpha^*\sigma_i^-).$$

Since the operators h_i commute, the partition function becomes:

$$Z = \frac{\exp[\![-\beta N\{|\alpha|^2 + K[2|\alpha|^2 + (\alpha)^2 + (\alpha^*)^2]\}]\!]}{1 - \exp[-\beta(2K+1)]} (\mathrm{Tr}\, e^{-\beta h_i})^N. \tag{7}$$

The operator h_i is a 2×2 matrix, in the standard spin-space representation, and can be diagonalized easily:

$$\begin{vmatrix} \tfrac{1}{2}\epsilon - \mu & \lambda\alpha \\ \lambda\alpha^* & -\tfrac{1}{2}\epsilon - \mu \end{vmatrix} = 0.$$

The two eigenvalues are:

$$\mu = \pm(\tfrac{1}{4}\epsilon^2 + \lambda^2 |\alpha|^2)^{1/2}. \tag{8}$$

The partition function can be now written as:

$$Z = \frac{\exp[\![-\beta N\{|\alpha|^2 + 2K[|\alpha|^2 + \tfrac{1}{2}(\alpha)^2 + \tfrac{1}{2}(\alpha^*)^2]\}]\!]}{1 - \exp[-\beta(2K+1)]} [2\cosh(\beta\mu)]^N. \tag{9}$$

The next problem is to determine the value of the parameter α, such that the terms on the right-hand side of equation (2) are minimized.

From the definitions of H_0 and H_1, we infer that:

$$\langle H_1 \rangle_0 = 0.$$

Therefore, α will be found from the condition:

$$\langle (H_1)^2 \rangle_0 = \text{minimum}. \tag{10}$$

Define the operator A:

$$A = N^{1/2}[\alpha^* + 2K(\alpha+\alpha^*)] + N^{-1/2}(\tfrac{1}{2}\lambda) \sum_{i=1}^{N} \sigma_i^+, \tag{11}$$

then $\langle (H_1)^2 \rangle_0$ can be written as:

$$\langle (H_1)^2 \rangle_0 = Z^{-1}\, \mathrm{Tr}[A^\dagger A(bb^\dagger + b^\dagger b) + K(b^2 b^{\dagger 2} + b^{\dagger 2} b^2)]\, e^{-\beta H_0}. \tag{12}$$

Since only the first term in the parenthesis is a function of α, we have to minimize the following:

$$Z^{-1}\,\mathrm{Tr}(bb^{\dagger}+b^{\dagger}b)\exp[-\beta b^{\dagger}b(2K+1)]\,\mathrm{Tr}(A^{\dagger}A)\exp\{-\beta[H_0-(2K+1)b^{\dagger}b]\}.$$

It is simple to prove that:

$$Z^{-1}\,\mathrm{Tr}(A^{\dagger}A)\exp\{-\beta[H_0-b^{\dagger}b(2K+1)]\}=\frac{2K}{\beta}+\frac{1}{\beta^2N}\left(\frac{1}{Z}\frac{\partial^2 Z}{\partial\alpha\,\partial\alpha^*}\right),$$

and since

$$Z^{-1}\left(\frac{\partial^2 Z}{\partial\alpha\,\partial\alpha^*}\right)=\left(\frac{\partial^2\ln Z}{\partial\alpha\,\partial\alpha^*}\right)+\left(\frac{\partial\ln Z}{\partial\alpha}\right)\left(\frac{\partial\ln Z}{\partial\alpha^*}\right),\tag{13}$$

we conclude that:

$$Z^{-1}\,\mathrm{Tr}(A^{\dagger}A)\exp\{-\beta[H_0-b^{\dagger}b(2K+1)]\}$$
$$=\frac{2K}{\beta}+\frac{1}{\beta}\left(\frac{\partial^2(\ln Z/N\beta)}{\partial\alpha\,\partial\alpha^*}+\frac{\partial(\ln Z/N\beta)}{\partial\alpha}\frac{\partial(\ln Z/N\beta)}{\partial\alpha^*}(N\beta)\right).\tag{14}$$

In order to minimize the second term of equation (14), we set:

$$\frac{1}{N\beta}\frac{\partial\ln Z}{\partial\alpha^*}=0,\tag{15}$$

and as a result, the following condition on α is found:

$$\alpha+4K\alpha_r=\left(\frac{\lambda^2}{2\mu}\right)\alpha\tanh(\beta\mu),\tag{16}$$

where:

$$\alpha=\alpha_r+\mathrm{i}\alpha_i.$$

From equation (16), the following cases are encountered.

(*a*) $\alpha_i=0$, $\alpha_r=0$, which corresponds to the 'normal state'.

(*b*) $\alpha_r\neq0$, $\alpha_i\neq0$. We cannot satisfy both conditions simultaneously, so this case is not a solution of equation (16).

(*c*) $\alpha_r=0$, $\alpha_i\neq0$. In this case the condition on α is:

$$\alpha\left(\frac{\lambda^2}{2\mu}\right)\tanh(\beta\mu)=\alpha,$$

and the free energy (per molecule) is:

$$f=|\alpha|^2-\beta^{-1}\ln[2\cosh(\beta\mu)].$$

(*d*) $\alpha_r\neq0$, $\alpha_i=0$. The condition on α is:

$$\alpha_r\left(\frac{\lambda^2}{2\mu}\right)\tanh(\beta\mu)=\alpha_r(4K+1),$$

and the free energy per molecule:

$$f=\alpha^2(1+4K)-\beta^{-1}\ln[2\cosh(\beta\mu)].$$

Turning now to the Dicke model with counter-rotating terms, the Hamiltonian can be written as:

$$H = a^\dagger a + \frac{\epsilon}{2}\sum_{i=1}^{N} \sigma_i^z + \frac{\lambda}{2\sqrt{N}}\sum_{i=1}^{N} [a(\sigma_i^+ + \sigma_i^-) + a^\dagger(\sigma_i^+ + \sigma_i^-)].$$

Following a procedure similar to the previous case, the partition function is found to be:

$$Z = \mathrm{Tr}[\exp(-\beta H_0)] = \frac{\exp(-\beta N|\alpha|^2)}{1-\exp(-\beta)}[2\cosh(\beta\mu)]^N, \tag{17}$$

where the eigenvalues (of h_i) are:

$$\mu = \pm(\tfrac{1}{4}\epsilon^2 + 4\alpha_r^2\lambda^2)^{1/2}. \tag{18}$$

The condition on α in this case turns out to be:

$$\alpha_r = (2\lambda^2/\mu)\alpha_r \tanh(\beta\mu), \qquad \alpha_i = 0. \tag{19}$$

The critical temperature, obtained by setting $\mu = \pm\frac{1}{2}\epsilon$, is:

$$\tanh\left(\frac{\beta_c\epsilon}{2}\right) = \frac{\epsilon}{4\lambda^2}. \tag{20}$$

These results are in agreement with previous authors (Carmichael *et al* 1973).

Summarizing the results, when the A^2 term is included in Dicke's Hamiltonian, there is a solution (case (c)) which is unchanged by the correction. An alternative proof leading to the same conclusion is given in Orszag to be published.

If the counter-rotating terms are included in the model, which is of interest particularly when the spin system interacts strongly with the field and the rotating-wave approximation is not valid, then the phase transition subsists, but the coupling constant is renormalized, namely:

$$\lambda^* = 2\lambda,$$

therefore, the critical temperature, the condition for critical behaviour and the free energy change accordingly.

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